

# NEGLIGIBLE VARIATION AND THE CHANGE OF VARIABLES THEOREM

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**ABSTRACT.** In this note we prove a necessary and sufficient condition for the change of variables formula for the HK integral, with implications for the change of variables formula for the Lebesgue integral. As a corollary, we obtain a necessary and sufficient condition for the Fundamental Theorem of Calculus to hold for the HK integral.

J. Serrin and D.E. Varberg [S&V] proved the following change of variables theorem for the Lebesgue integral.

**Theorem 1.** *Assume that  $g : [a, b] \rightarrow \mathbb{R}$  is differentiable<sup>1</sup> almost everywhere and that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is Lebesgue integrable on  $[c, d] \supseteq g([a, b])$ . Then  $(f \circ g) \cdot g'$  is Lebesgue integrable on  $[a, b]$  and the change of variables formula*

$$(1) \quad \int_{g(\alpha)}^{g(\beta)} f(u) du = \int_{\alpha}^{\beta} f(g(s)) g'(s) ds$$

*holds for all  $\alpha, \beta$  in  $[a, b]$  if and only if  $F \circ g$  is absolutely continuous, where  $F(x) := \int_{g(a)}^x f(u) du$ .*

K. Krzyzewski [Krz1] and G. Goodman [Goodman] proved several sufficient but not necessary conditions for (1) to hold for the Denjoy and Perron integrals. Both integrals are equivalent to Henstock-Kurzweil (HK) integral, for which we establish necessary and sufficient conditions for the change of variables theorem. As a consequence, we obtain the optimal condition for which (1) holds for a fixed  $\alpha, \beta$  without the requirement that it holds for every subinterval. Furthermore, we show that even when  $\int_{g(\alpha)}^{g(\beta)} f(u) du$  is a Lebesgue integral, (1) holds under weaker conditions than those of Theorem 1.

## PRELIMINARIES

For excellent presentations of the HK integral, see [Bartle], [Lee & Vyborny], and [Gordon]. For the reader's convenience, we include the basic definitions necessary to follow the exposition below.

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2010 *Mathematics Subject Classification.* Primary 26A36, 26A39, 26A46; Secondary 26A30.

*Key words and phrases.* change of variables, Henstock-Kurzweil integral, absolute continuity, negligible variation.

<sup>1</sup>Unless otherwise noted, we use differentiable to mean finitely differentiable.

We denote the closed interval  $[a_j, b_j]$  as  $I_j$  and  $|I_j| = b_j - a_j$ .

**Definition 1.** A *tagged partition*  $P$  of  $[a, b]$ , denoted  $P[a, b]$ , is a finite set of the form  $\{(x_j, I_j) : 1 \leq j \leq n\}$  such that  $\bigcup_{j=1}^n I_j = [a, b]$ ,  $x_j \in I_j$  for all  $j$ , and  $i \neq j$  implies that  $(a_i, b_i) \cap (a_j, b_j) = \emptyset$ .

**Definition 2.** A *gauge* for the interval  $[a, b]$  is a function from  $[a, b]$  to the positive real numbers,  $\mathbb{R}^+$ .

**Definition 3.** A tagged partition  $P[a, b] = \{(x_j, I_j) : 1 \leq j \leq n\}$  is said to be *subordinate* to the gauge  $\delta$  if  $I_j \subseteq (x_j - \delta(x_j), x_j + \delta(x_j))$  for every  $j$ .

**Definition 4.** Given  $f : [a, b] \rightarrow \mathbb{R}$  and a tagged partition  $P[a, b] = \{(x_j, I_j) : 1 \leq j \leq n\}$ ,  $R_P f := \sum_{j=1}^n f(x_j) \cdot |I_j|$  is called their *Riemann sum*.

**Definition 5.** A function  $f : [a, b] \rightarrow \mathbb{R}$  is said to be *HK integrable* over  $[a, b]$  if there exists a number,  $(HK) \int_a^b f(x) dx$ , so that for any  $\epsilon > 0$  there exists a gauge  $\delta$  on  $[a, b]$  such that any tagged partition  $P[a, b]$  subordinate to  $\delta$  satisfies  $|R_P f - (HK) \int_a^b f(x) dx| < \epsilon$ . We also define  $(HK) \int_b^a f(x) dx = -(HK) \int_a^b f(x) dx$ .

Since two integrands which differ only on a set of measure zero have the same HK integral [Lee & Vyborny, Theorem 2.5.6], we adopt the convention that, given a function  $f$  that is defined almost everywhere, its HK integral is the integral of the function that is equal to  $f$  where  $f$  is defined, and is 0 where  $f$  is not defined.

**Definition 6.** A function  $f$  has *negligible variation* on a set  $E \subseteq [a, b]$  if for any  $\epsilon > 0$  there exists a gauge  $\delta$  on  $[a, b]$  such that for any tagged partition  $P[a, b] = \{(x_j, I_j) : 1 \leq j \leq n\}$  subordinate to  $\delta$ ,  $\sum_{x_j \in E} |f(b_j) - f(a_j)| < \epsilon$ .

From this point on, we will denote  $\Delta_j f = f(b_j) - f(a_j)$ .

The definition of negligible variation and a broad range of applications was introduced by [Vyborny]. One of them is the following theorem, which was proven in the following necessary and sufficient form in Theorem 5.12 of [Bartle].

**Theorem 2** (Fundamental Theorem of Calculus for the HK Integral).

$F(x) - F(a) = (HK) \int_a^x f(s) ds$  for all  $x \in [a, b]$  if and only if there exists a set  $E \subseteq [a, b]$  such that  $F'(x) = f(x)$  for all  $x \in E$  and  $[a, b] \setminus E$  is a set of measure zero on which  $F$  has negligible variation.

In the context of the HK integral, negligible variation plays a role analogous to that played by absolute continuity in the context of the Lebesgue integral. Corollary 14.8 of [Bartle] proved the following relationship between the two.

**Theorem 3.**  $F$  is absolutely continuous on  $[a, b]$  if and only if  $F$  is of bounded variation on  $[a, b]$  and  $F$  has negligible variation on every subset of  $[a, b]$  that has measure zero.

## 1. CHANGE OF VARIABLES ON A SINGLE INTERVAL

**Definition 7.** A function  $f$  has *negligible conditional variation* on a set  $E \subseteq [a, b]$  if for any  $\epsilon > 0$  there exists a gauge  $\delta$  on  $[a, b]$  such that for any tagged partition  $P[a, b] = \{(x_j, I_j) : 1 \leq j \leq n\}$  subordinate to  $\delta$ ,  $\left| \sum_{x_j \in E} \Delta_j f \right| < \epsilon$ .

Some functions may have negligible conditional variation but not negligible variation on the set of points where they fail to be differentiable; a simple example is the indicator function of an open interval contained in  $[a, b]$ . We will present a continuous function with this property in Example 1 of Section 3.

The same examples show that, although a function that has negligible variation on a set also has negligible variation on all its subsets, this is not true for negligible conditional variation.

We will need the following theorem, which was proven in [Krz1] and [S&V]. For a stronger version of the theorem, see Theorem 7.

**Theorem 4.** *If  $g$  has a derivative (finite or infinite) on a set  $E$  and  $g(E)$  has measure zero, then  $g' = 0$  almost everywhere on  $E$ .*

**Lemma 1.** *Assume that both  $g : [a, b] \rightarrow D$  and  $F : D \rightarrow \mathbb{R}$  have derivatives almost everywhere and that  $f = F'$  almost everywhere. Then  $g'(x) = 0$  at almost every  $x \in [a, b]$  where the equality*

$$(2) \quad (F \circ g)'(x) = (f \circ g \cdot g')(x)$$

*fails, that is to say where (2) is false or either side is undefined.*

*Proof.* Let  $Z$  be the null set where  $F$  does not have a derivative equal to  $f$ . By Theorem 4,  $g'(x) = 0$  for almost every  $x \in g^{-1}(Z)$ . On the complement of  $g^{-1}(Z)$ , (2) holds at all  $x$  where  $g'(x)$  exists, and so almost everywhere.  $\square$

**Theorem 5.** *Assume that  $g : [a, b] \rightarrow \mathbb{R}$  has a derivative almost everywhere and that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is HK integrable on every interval with endpoints in the range of  $g$ .<sup>2</sup> Define  $F(x) := (HK) \int_{g(a)}^x f(u) du$ . Then  $(f \circ g) \cdot g'$  is HK integrable on  $[a, b]$  and the change of variables formula*

$$(HK) \int_{g(a)}^{g(b)} f(u) du = (HK) \int_a^b f(g(s)) g'(s) ds$$

*holds if and only if  $F \circ g$  has negligible conditional variation on the set where  $(F \circ g)' = f \circ g \cdot g'$  fails.*

<sup>2</sup>If the HK integral of  $f$  exists on an interval then it also exists on every subinterval [Bartle, Corollary 3.8]. It is therefore sufficient to require that  $f$  be HK integrable over an interval containing the range of  $g$ .

*Proof.* Let  $B$  be the set where  $(F \circ g)' = f \circ g \cdot g'$  fails and assume  $F \circ g$  has negligible conditional variation there. Let  $h(x) = 0$  if  $x \in B$  and  $h(x) = g'(x)$  otherwise.

By Theorem 2,  $F$  has a derivative equal to  $f$  almost everywhere and so by Lemma 1,  $g' = h = 0$  almost everywhere on  $B$ . Consequently,

$$(HK) \int_a^b (f \circ g \cdot g')(s) ds = (HK) \int_a^b (f \circ g \cdot h)(s) ds.$$

Since  $F \circ g$  has a derivative on the complement of  $B$ , there exists for any  $\epsilon > 0$  a function  $\eta_\epsilon : [a, b] \setminus B \rightarrow \mathbb{R}^+$  such that if  $y \in [x - \eta_\epsilon(x), x + \eta_\epsilon(x)] \cap [a, b]$  then

$$|(F \circ g)'(x) \cdot (y - x) - ((F \circ g)(y) - (F \circ g)(x))| < \epsilon |y - x| / (b - a)$$

Also, because  $F \circ g$  has negligible conditional variation on  $B$ , there exists a gauge  $\delta_1$  on  $[a, b]$  such that for any tagged partition  $P[a, b] = \{(x_j, I_j) : 1 \leq j \leq n\}$  subordinate to  $\delta_1$ ,  $|\sum_{x_j \in B} \Delta_j f| < \epsilon/2$ . Let  $\delta$  be a gauge on  $[a, b]$  so that  $\delta(x) = \eta_{\epsilon/2}(x)$  if  $x \notin B$  and  $\delta(x) = \delta_1(x)$  if  $x \in B$ .

Consider a tagged partition  $P[a, b] = \{(x_j, I_j) : 1 \leq j \leq n\}$  subordinate to  $\delta$ . The Riemann sum of  $f \circ g \cdot h$  corresponding to this tagged partition is

$$\begin{aligned} R_P f \circ g \cdot h &= \overbrace{\sum_{x_j \in B} (f \circ g \cdot h)(x_j) \cdot |I_j|}^{=0} + \sum_{x_j \notin B} (f \circ g \cdot h)(x_j) \cdot |I_j| \\ &= \sum_{x_j \notin B} (F \circ g)'(x_j) \cdot |I_j| \\ &= \sum_{x_j \notin B} [(F \circ g)'(x_j) \cdot |I_j| - \Delta_j(F \circ g)] \\ &\quad + \sum_{x_j \notin B} \Delta_j(F \circ g) \\ &= \sum_{x_j \notin B} [(F \circ g)'(x_j) \cdot |I_j| - \Delta_j(F \circ g)] \\ &\quad + \sum_{x_j \in [a, b]} \Delta_j(F \circ g) - \sum_{x_j \in B} \Delta_j(F \circ g) \\ &= \sum_{x_j \notin B} [(F \circ g)'(x_j) \cdot |I_j| - \Delta_j(F \circ g)] \\ &\quad + (HK) \int_{g(a)}^{g(b)} f(u) du - \sum_{x_j \in B} \Delta_j(F \circ g). \end{aligned}$$

And so

$$\begin{aligned} (3) \quad R_P f \circ g \cdot h - (HK) \int_{g(a)}^{g(b)} f(u) du \\ = \sum_{x_j \notin B} [(F \circ g)'(x_j) \cdot |I_j| - \Delta_j(F \circ g)] - \sum_{x_j \in B} \Delta_j(F \circ g). \end{aligned}$$

Since  $F \circ g$  has negligible conditional variation on  $B$  and  $\delta$  is chosen accordingly,

$$\left| \sum_{x_j \in B} \Delta_j (F \circ g) \right| < \epsilon/2.$$

Also, for any  $x_j \notin B$ ,

$$|(F \circ g)'(x_j) \cdot |I_j| - \Delta_j (F \circ g)| < \epsilon |I_j| / 2(b-a),$$

and so

$$\left| \sum_{x_j \notin B} (F \circ g)'(x_j) \cdot |I_j| - \Delta_j (F \circ g) \right| < \epsilon/2.$$

Therefore

$$\left| R_P f \circ g \cdot h - (HK) \int_{g(a)}^{g(b)} f(u) du \right| < \epsilon,$$

proving that  $(HK) \int_a^b f(g(s)) g'(s) ds$  exists and is equal to  $(HK) \int_{g(a)}^{g(b)} f(u) du$ .

Conversely, choose  $\epsilon > 0$ , let  $B$ ,  $h$ , and  $\eta_\epsilon$  be defined as above, and assume  $(HK) \int_{g(a)}^{g(b)} f(u) du = (HK) \int_a^b f(g(s)) h(s) ds$ . Thus there exists a gauge  $\delta_1$  on  $[a, b]$  so that for any tagged partition  $P$  subordinate to  $\delta_1$ ,

$$(4) \quad \left| R_P f \circ g \cdot h - (HK) \int_{g(a)}^{g(b)} f(u) du \right| < \epsilon/2.$$

Let  $\delta$  be a gauge on  $[a, b]$  so that  $\delta(x) = \min \{\delta_1(x), \eta_{\epsilon/2}(x)\}$  if  $x \notin B$ , and  $\delta(x) = \delta_1(x)$  if  $x \in B$ .

Choose any tagged partition  $P[a, b] = \{(x_j, [a_j, b_j]) : 1 \leq j \leq n\}$  subordinate to  $\delta$ . Consequently it is also subordinate to  $\delta_1$ , and so (4) holds. By (3),

$$\left| \sum_{x_j \notin B} (F \circ g)'(x_j) \cdot |I_j| - \Delta_j (F \circ g) - \sum_{x_j \in B} \Delta_j (F \circ g) \right| < \epsilon/2.$$

Also,

$$\left| \sum_{x_j \notin B} (F \circ g)'(x_j) \cdot |I_j| - \Delta_j (F \circ g) \right| < \epsilon/2.$$

Therefore

$$\left| \sum_{x_j \in B} \Delta_j (F \circ g) \right| < \epsilon,$$

proving that  $F \circ g$  has negligible conditional variation on  $B$ .  $\square$

By taking  $F(x) = x$ , we obtain as a corollary the following necessary and sufficient condition for the Fundamental Theorem of Calculus for the HK integral to hold for a particular interval, rather than all subintervals.

**Corollary 1.** *Assume that  $g : [a, b] \rightarrow \mathbb{R}$  is differentiable almost everywhere on  $[a, b]$ . Then  $g'$  is HK integrable on  $[a, b]$  and  $g(b) - g(a) = (HK) \int_a^b g'(s) ds$  if and only if  $g$  has negligible conditional variation on the set where it is not differentiable.*

Theorem 5 complements Theorem 1 by obtaining a necessary and sufficient condition for change of variables to hold on a single interval. However, even when one side of (1) is a Lebesgue integral, the integral on the other side sometimes must be taken in the HK sense. For example, take  $g$  as any function that is not an indefinite Lebesgue integral but which satisfies the condition of Corollary 1 and let  $f(x) = 1$ .

## 2. CHANGE OF VARIABLES ON ALL SUBINTERVALS

The following recasting of the Saks-Henstock Lemma for negligible variation provides a corollary to Theorem 5 where change of variables holds for each subinterval and, in this sense, provides the precise HK analog of the Serrin and Varberg theorem.

**Lemma 2.** *Let  $f$  be a real-valued function on  $[a, b]$  and  $E$  a subset of  $[a, b]$ . Assume that  $f$  has negligible conditional variation on  $E \cap [\alpha, \beta]$  for every  $[\alpha, \beta] \subseteq [a, b]$ . Then  $f$  has negligible variation on  $E$ .*

*Proof.* Choose  $\epsilon > 0$  and let  $\delta$  be a gauge on  $[a, b]$  such that for any tagged partition  $P[a, b] = \{(x_j, I_j) : 1 \leq j \leq n\}$  subordinate to  $\delta$ ,

$$\left| \sum_{x_j \in E} \Delta_j f \right| < \epsilon$$

Fix such a tagged partition  $P$  and choose  $\epsilon' > 0$ . Since  $f$  has negligible conditional variation on each  $I_j$ , there exists a gauge,  $\delta_j$ , such that if  $\{(x_{j,k}, I_{x_{j,k}}) : 1 \leq k \leq n_j\}$  is a tagged partition of  $[a_j, b_j]$  subordinate to  $\delta_j$ , then

$$\left| \sum_{x_{j,k} \in E} \Delta_{j,k} f \right| < \epsilon'/n.$$

Let  $Q_j = \{(x_{j,k}, I_{x_{j,k}}) : 1 \leq k \leq n_j\}$  be a tagged partition of  $I_j$  subordinate to  $\min\{\delta, \delta_j\}$ .

Also let  $L$  be the subset of  $\{1 \dots n\}$  such that if  $j \in L$  then  $\Delta_j f \geq 0$ . Now let

$$R = \{(x_j, I_j) : j \in L\} \cup \left( \bigcup_{j \notin L} Q_j \right).$$

Since  $R$  is a tagged partition of  $[a, b]$  subordinate to  $\delta$ ,

$$\epsilon > \left| \sum_{\substack{(x, [\alpha, \beta]) \in R \\ x \in E}} f(\beta) - f(\alpha) \right| = \left| \sum_{\substack{j \in L \\ x_j \in E}} \Delta_j f + \sum_{j \notin L} \sum_{x_{j,k} \in E} \Delta_{j,k} f \right|.$$

Also, because each  $Q_j$  is subordinate to  $\delta_j$ ,

$$\epsilon' > \sum_{j \notin L} \left| \sum_{x_{j,k} \in E} \Delta_{j,k} f \right| \geq \left| \sum_{j \notin L} \sum_{x_{j,k} \in E} \Delta_{j,k} f \right|.$$

Consequently,

$$\epsilon + \epsilon' > \left| \sum_{\substack{j \in L \\ x_j \in E}} \Delta_j f \right| = \sum_{\substack{j \in L \\ x_j \in E}} |\Delta_j f|.$$

Since the choice of  $\epsilon' > 0$  was arbitrary, it must be true that

$$\epsilon \geq \sum_{\substack{j \in L \\ x_j \in E}} |\Delta_j f|.$$

Similarly,

$$\epsilon \geq \sum_{\substack{j \notin L \\ x_j \in E}} |\Delta_j f|.$$

Therefore

$$2\epsilon \geq \sum_{x_j \in E} |\Delta_j f|,$$

proving that  $f$  has negligible variation on  $E$ .  $\square$

**Corollary 2.** *Assume that  $g : [a, b] \rightarrow \mathbb{R}$  is differentiable almost everywhere and that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is HK integrable on every interval with endpoints in the range of  $g$ . Define  $F(x) := (HK) \int_{g(a)}^x f(u) du$ . Then  $(f \circ g) \cdot g'$  is HK integrable on  $[a, b]$  and the change of variables formula*

$$(HK) \int_{g(\alpha)}^{g(\beta)} f(u) du = (HK) \int_{\alpha}^{\beta} f(g(s)) g'(s) ds$$

*holds for every  $[\alpha, \beta] \subseteq [a, b]$  if and only if  $F \circ g$  has negligible variation on the set where  $(F \circ g)' = f \circ g \cdot g'$  fails.*

The necessary and sufficient condition that  $F \circ g$  have negligible variation on the set where  $(F \circ g)' = f \circ g \cdot g'$  fails is clearly justified by the preceding lemma; in Theorem 6 of the next section, we prove that it is equivalent to the condition that  $F \circ g$  have negligible variation on each null set and on the set where  $g' = 0$ .

*Remark 1.* A tempting possibility to investigate is whether HK integrals automatically satisfy the requirements of the substituting function in the change of variables formula. In other words, if  $g$  is an indefinite HK integral, is it true that for any HK integrable  $f$  that  $(HK) \int_{g(\alpha)}^{g(\beta)} f(u) du = (HK) \int_{\alpha}^{\beta} f(g(s)) g'(s) ds$ ? If this were true, then the composition of two indefinite HK integrals would be an indefinite HK integral. However, this is not the case, as the following two indefinite Lebesgue integrals (and hence also HK integrals) fail to have a composition which is an HK integral.

Let  $S$  be the Smith–Volterra–Cantor set of measure  $\frac{1}{2}$  [Bressoud, pp. 90-91] constructed on a unit interval through the usual method of deleting an open interval of length  $4^{-n}$  from the center of each of the  $2^{n-1}$  intervals at step  $n$ , leaving a closed nowhere-dense set of positive measure in the limit. Let  $G(x) = \text{dist}(x, S)$  and  $F(x) = \sqrt[4]{x}$ . Consequently  $F$  and  $G$  are both indefinite Riemann integrals, yet for any  $x \in S$  and  $n \in \mathbb{N}$  there exists  $y \in (x - 2^{-n}, x + 2^{-n})$  such that  $(y - 2^{-2n-3}, y + 2^{-2n-3}) \subseteq S^c$ . Thus  $\left| \frac{F(G(y)) - F(G(x))}{y - x} \right| = \frac{F(G(y))}{|y - x|} > \frac{\sqrt[4]{2^{-2n-3}}}{2^{-n}} = \sqrt[4]{2^{2n-3}}$ . Therefore  $F \circ G$  has no derivative on  $S$ , a set of positive measure, and so cannot be the indefinite HK integral of any function.

Note that  $G$  is differentiable except on the endpoints and midpoint of each deleted interval, which is a countable set, and that the construction of  $G$  could be altered so that those points are differentiable as well, with the same result.

### 3. NEGLIGIBLE VARIATION

**Example 1.** The following is a continuous function that has negligible conditional variation and not negligible variation on a set.

Let  $C$  denote the Cantor set and  $c$  the Cantor-Lebesgue function on  $[0, 1]$ . Let  $D = C \cup (-C)$ . Let us denote

$$\delta_D(x) = \begin{cases} 1 & \text{if } x \in D \\ \text{dist}(x, D) & \text{if } x \notin D \end{cases}.$$

If  $\{(x_j, I_j) : 1 \leq j \leq n\}$  is a tagged partition of  $[-1, 1]$  subordinate to  $\delta_D$ , then

$$\begin{aligned} 0 &= |c(|1|) - c(|-1|)| \\ &= \left| \sum_{x_j \in D} \Delta_j c(|\cdot|) + \sum_{x_j \notin D} \overbrace{\Delta_j c(|\cdot|)}^{=0} \right|. \end{aligned}$$

Thus  $c(|\cdot|)$  has negligible conditional variation on  $D$ .

Suppose there exists  $\delta$  so that for  $P[-1, 1] = \{(x_j, I_j) : 1 \leq j \leq n\}$  subordinate to  $\delta$ ,  $\sum_{x_j \in D} |\Delta_j c(|\cdot|)| < 1$ . Form a tagged partition  $P[-1, 1] = \{(x_j, I_j) : 1 \leq j \leq n\}$  as a union of a  $Q_1[-1, 0]$  and  $Q_2[-1, 0]$  subordinate to  $\min\{\delta_D(x), \delta_1(x)\}$ . These



three partitions are a fortiori partitions subordinate to  $\delta$ . Then, as above,

$$\begin{aligned}
 2 &= |c(|-1|) - c(|0|)| + |c(|0|) - c(|1|)| \\
 &= \sum_{\substack{x_j \in D \\ x_j \in [-1,0]}} |\Delta_j c(|\cdot|)| + \sum_{\substack{x_j \notin D \\ x_j \in [-1,0]}} \overbrace{|\Delta_j c(|\cdot|)|}^{=0} \\
 &\quad + \sum_{\substack{x_j \in D \\ x_j \in [0,1]}} |\Delta_j c(|\cdot|)| + \sum_{\substack{x_j \notin D \\ x_j \in [0,1]}} \overbrace{|\Delta_j c(|\cdot|)|}^{=0}.
 \end{aligned}$$

Thus  $c(|\cdot|)$  does not have negligible variation on  $D$ . This argument also shows that  $c(|\cdot|)$  does not have negligible conditional variation on  $D \cap [0, 1]$ .

### 3.1. Conditions That Imply Negligible Variation.

**Lemma 3.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f'(x) = 0 \forall x \in D \subseteq [a, b]$ . Then  $f$  has negligible variation on  $D$ .*

*Proof.* Choose  $\epsilon > 0$ . Let  $\eta_\epsilon : D \rightarrow \mathbb{R}^+$  be a function such that if  $y \in [x - \eta_\epsilon(x), x + \eta_\epsilon(x)] \cap [a, b]$  then

$$|f(y) - f(x)| \leq \epsilon |y - x| / (b - a)$$

and let

$$\delta(x) = \begin{cases} \eta_\epsilon(x) & \text{if } x \in D \\ 1 & \text{if } x \notin D \end{cases}$$

Choose a tagged partition  $P[a, b] = \{(x_j, I_j) : 1 \leq j \leq n\}$  subordinate to  $\delta$ . Then

$$\sum_{x_j \in D} |\Delta_j f| \leq \left( \frac{\epsilon}{b - a} \right) \sum_{x_j \in D} |I_j| \leq \epsilon.$$

□

**Lemma 4.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  have finite upper and lower Dini derivatives on a null set  $Z$ ; that is to say  $\overline{D}f(x) := \limsup_{y \rightarrow x} \left| \frac{f(y) - f(x)}{y - x} \right| < \infty$  for all  $x \in Z$ . Then  $f$  has negligible variation on  $Z$ .*

*Proof.* Let  $\epsilon > 0$  and let  $Z_n = Z \cap \{x : \overline{D}f(x) \in [n, n + 1)\}$ . Also, let  $\eta_1 : Z \rightarrow \mathbb{R}^+$  be a function such that if  $y \in [x - \eta_1(x), x + \eta_1(x)] \cap [a, b]$  then

$$|f(y) - f(x)| \leq (1 + \lfloor \overline{D}f(x) \rfloor) |y - x|$$

Let  $C_n \supseteq Z_n$  be open sets with measure less than  $\frac{\epsilon}{2^{n+1}(n+2)}$ . Define

$$\delta(x) = \begin{cases} \min \{ \eta_1(x), \text{dist}(x, C_n^c) \} & \text{if } x \in Z_n \\ 1 & \text{if } x \notin Z \end{cases}$$

Choose a tagged partition  $P[a, b] = \{(x_j, I_j) : 1 \leq j \leq n\}$  subordinate to  $\delta$ . If  $x_j \in Z_n$  for some  $j$ , then  $|\overline{D}f(x_j)| \in [n, n+1)$ , so

$$|\Delta_j f| \leq |f(b_j) - f(x_j)| + |f(x_j) - f(a_j)| \leq (n+2) \cdot |I_j|.$$

Therefore

$$\sum_{x_j \in Z} |\Delta_j f| = \sum_{n=0}^{\infty} \sum_{x_j \in Z_n} |\Delta_j f| \leq \sum_{n=0}^{\infty} (n+2) \cdot \lambda(C_n) \leq \epsilon,$$

proving that  $f$  has negligible variation on  $Z$ .  $\square$

Clearly, if a function has negligible variation on a set  $N$ , then it has negligible conditional variation on  $S \supseteq N$  if and only if it has negligible conditional variation on  $S \setminus N$ .

Consider Theorem 5, where  $B$  is the set where  $(F \circ g)' = f \circ g \cdot g'$  fails. Let  $A_F$  and  $A_g$  be the sets where  $F$  and  $g$  fail to have derivatives and set  $A = g^{-1}(A_F) \cup A_g$ . By Lemma 1,  $g'(x) = 0$  for almost every  $x \in B$ . Similarly,  $A_F$  and  $A_g$  have measure zero and  $g'$  is defined almost everywhere, so  $g'(x) = 0$  at almost every  $x \in A$ . Furthermore,

$$(F \circ g)'(x) = \begin{cases} (F' \circ g \cdot g')(x) & \text{if } x \in B \setminus A \\ (f \circ g \cdot g')(x) & \text{if } x \in A \setminus B \end{cases}$$

so  $(F \circ g)'$  is zero almost everywhere on  $(B \setminus A) \cup (A \setminus B)$ . By Lemma 3,  $F \circ g$  has negligible variation on  $(B \setminus A) \cup (A \setminus B)$ . This proves that if  $F \circ g$  has negligible conditional variation on any set  $S$  such that  $B \cap A \subseteq S \subseteq B \cup A$ , then it has it on every other such set.

**Theorem 6.** *Assume that  $g : [a, b] \rightarrow D$  and  $F : D \rightarrow \mathbb{R}$  have derivatives almost everywhere and that  $f = F'$  almost everywhere. Then  $F \circ g$  has negligible variation on the set where  $(F \circ g)' = f \circ g \cdot g'$  fails if and only if  $F \circ g$  has negligible variation on each null set and on the set where  $g'$  is zero.*

*Proof.* Let  $B$  again be the set where  $(F \circ g)' = f \circ g \cdot g'$  fails and assume  $F \circ g$  has negligible variation on  $B$ . It then has negligible variation on all subsets of  $B$  as well, including  $B \cap \{x : g'(x) = 0\}$ . Since  $(F \circ g)' = f \circ g \cdot g'$  on the complement of  $B$ , then, by Lemma 3,  $F \circ g$  has negligible variation on  $\{x : g'(x) = 0\} \setminus B$ . Therefore  $F \circ g$  has negligible variation on the set  $\{x : g'(x) = 0\}$ .

Similarly, for any null set  $Z$ ,  $F \circ g$  will have negligible variation on  $Z \cap B$ , since that is a subset of  $B$ . Also, by Lemma 4,  $F \circ g$  will have negligible variation on  $Z \setminus B$ . Therefore  $F \circ g$  has negligible variation on  $Z$ .

Conversely, by Lemma 1, there exists a null set  $Z$  and a set  $E \subseteq \{x : g'(x) = 0\}$  such that  $B = Z \cup E$ . Consequently if  $F \circ g$  has negligible variation on each null set, it must have it on  $Z$  in particular. Also, if it has negligible variation on

$\{x : g'(x) = 0\}$ , then it has it on its subsets such as  $E$ . Therefore  $F \circ g$  has negligible variation on  $B$ .  $\square$

We may therefore restate Corollary 2 in the following equivalent form.

**Corollary 3.** *Assume that  $g : [a, b] \rightarrow \mathbb{R}$  is differentiable almost everywhere and that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is HK integrable on every interval with endpoints in the range of  $g$ . Define  $F(x) := (HK) \int_{g(a)}^x f(u) du$ . Then  $(f \circ g) \cdot g'$  is HK integrable on  $[a, b]$  and the change of variables formula*

$$(HK) \int_{g(\alpha)}^{g(\beta)} f(u) du = (HK) \int_{\alpha}^{\beta} f(g(s)) g'(s) ds$$

*holds for every  $[\alpha, \beta] \subseteq [a, b]$  if and only if  $F \circ g$  has negligible variation on each null set and on the set where  $g'$  is zero.*

**3.2. Implications of Negligible Variation.** Functions that satisfy the conditions of Theorem 4 on a set  $E$  have a derivative equal to zero almost everywhere on  $E$ , so it follows from Lemma 3 that these functions have negligible variation on a subset of  $E$  of full measure. We show next that the conclusion of Theorem 4 holds for this larger class of functions.

**Theorem 7.** *If  $g : [a, b] \rightarrow \mathbb{R}$  has negligible variation on  $E \subseteq [a, b]$ , then  $\overline{D}g(x) = 0$  almost everywhere on  $E$ .*

*Proof.* Let  $\overline{D}_+g(x) := \limsup_{h \rightarrow 0+} \left| \frac{g(x+h) - g(x)}{h} \right|$  and  $E_{\gamma} = \{x \in E \setminus \{b\} : \overline{D}_+g(x) > \gamma\}$ . For reasons of symmetry, it suffices to show that  $\lambda(E_0) = 0$ . Furthermore,  $\lambda^*(E_0) \leq \sum_{n=1}^{\infty} \lambda^*\left(E_{\frac{1}{n}}\right)$ , so it is sufficient to show  $\lambda^*(E_{\gamma}) = 0$  for every  $\gamma > 0$ .

Choose  $\gamma, \epsilon > 0$ . Let  $\delta$  be a gauge such that for any tagged partition  $P[a, b] = \{(x_j, I_j) : 1 \leq j \leq n\}$  subordinate to  $\delta$ ,  $\sum_{x_j \in E} |\Delta_j g| < \epsilon\gamma/4$ . Let

$$C = \left\{ [\alpha, \beta] \subseteq [a, b] : 0 < \beta - \alpha < \delta(\alpha) \text{ and } \alpha \in E_{\gamma} \text{ and } \left| \frac{g(\beta) - g(\alpha)}{\beta - \alpha} \right| > \gamma/2 \right\}.$$

Since  $C$  is a Vitali cover of  $E_{\gamma}$ , there is a finite collection  $D$  of disjoint intervals in  $C$  such that  $\lambda^*(E_{\gamma} \setminus \bigcup D) < \epsilon/2$ .

Because  $D$  is a finite collection of closed intervals, there exists a finite collection  $O$  of disjoint open (in  $[a, b]$ ) intervals complementing  $D$ . By Cousin's Lemma, for each  $(\alpha, \beta) \in O$  there exists a tagged partition  $Q[\alpha, \beta]$  subordinate to  $\delta$ . Let

$$P[a, b] = \{(\alpha, [\alpha, \beta]) : [\alpha, \beta] \in D\} \cup \left( \bigcup_{(\alpha, \beta) \in O} Q[\alpha, \beta] \right).$$

Hence  $P[a, b]$  is a tagged partition  $\{(x_j, I_j) : 1 \leq j \leq n\}$  subordinate to  $\delta$ . So

$$\epsilon\gamma/4 > \sum_{x_j \in E} |\Delta_j g| \geq \sum_{I_j \in D} |\Delta_j g| \geq \sum_{I_j \in D} |I_j| \cdot \gamma/2.$$

Therefore  $\lambda(\bigcup D) = \sum_{I_j \in D} |I_j| < \epsilon/2$  and so  $\lambda^*(E_\gamma) \leq \lambda(\bigcup D) + \lambda^*(E_\gamma \setminus \bigcup D) < \epsilon$ . Since  $\epsilon$  was arbitrary,  $0 = \lambda(E_\gamma)$ .  $\square$

Theorem 7 and Lemma 3 show that  $f : [a, b] \rightarrow \mathbb{R}$  has negligible variation on a set  $E \subseteq [a, b]$  if and only if there exists a null set  $Z \subseteq E$  such that  $f$  has negligible variation on  $Z$  and  $f'(x) = 0$  for all  $x \in E \setminus Z$ .

**Theorem 8.** *If  $g : [a, b] \rightarrow \mathbb{R}$  has negligible variation on  $E \subseteq [a, b]$ , then  $\lambda(g(E)) = 0$ .*

*Proof.* We can clearly assume that  $a, b \notin E$ .

Choose  $\epsilon > 0$  and let  $\delta$  be a gauge such that for any tagged partition  $P[a, b] = \{(x_j, I_j) : 1 \leq j \leq m\}$  subordinate to  $\delta$ ,  $\sum_{x_j \in E} |\Delta_j g| < \epsilon$ . Let  $\eta_x = \min\{b - x, x - a, \delta(x)\}$ . Thus for  $x \in E$

$$\sup_{|h| \leq \eta_x} |g(x + h) - g(x)| \leq \epsilon,$$

and we denote this finite-valued supremum as  $s(x)$ . For every  $x \in E$ , choose  $h_x$  so that  $|g(x + h_x) - g(x)| \geq s(x)/2$  and  $|h_x| \leq \eta_x$ .

Define  $C(T) = \{(x - |h_x|, x + |h_x|) : x \in T\}$ .  $C(E)$  is a Besicovitch cover of  $E$ , so there exist two sequences<sup>3</sup> (possibly finite) of distinct points from  $E$ ,  $\{y_i\}$  and  $\{z_i\}$ , such that  $C(\{y_i\}), C(\{z_i\})$  each consist of disjoint intervals and  $C(\{y_i\}) \cup C(\{z_i\})$  covers  $E$ .

Since the closure of  $C(\{y_i\}_{i=1}^n)$  is a finite union of closed intervals, there exists a finite collection  $O_n$  of disjoint open (in  $[a, b]$ ) intervals complementing it. Also, for each  $(\alpha, \beta) \subseteq [a, b]$  there exists  $Q[\alpha, \beta]$  subordinate to  $\delta$ . Let

$$\begin{aligned} P_n[a, b] &= \left( \bigcup_{i=1}^n \{(y_i, [y_i - |h_x|, y_i]), (y_i, [y_i, y_i + |h_x|])\} \right) \\ &\cup \left( \bigcup_{(\alpha, \beta) \in O_n} Q[\alpha, \beta] \right). \end{aligned}$$

Hence  $P_n[a, b]$  is a tagged partition subordinate to  $\delta$  and so

$$\epsilon > \sum_{i=1}^n |g(y_i + h_x) - g(y_i)| \geq \sum_{i=1}^n s(y_i)/2.$$

Define  $D_n = \bigcup_{i=1}^n [g(y_i) - s(y_i), g(y_i) + s(y_i)]$ . Then  $D_{n+1} \supseteq D_n$  and, from the inequality above,  $4\epsilon > \lambda(D_n)$  for all  $n$ . Additionally,  $\bigcup_{n=1}^\infty D_n \supseteq g(C(\{y_i\}))$ . Thus  $4\epsilon \geq \lambda(\bigcup_{n=1}^\infty D_n) \geq \lambda^*(g(C(\{y_i\})))$ . The same argument applies for  $\{z_i\}$ , so

$$8\epsilon \geq \lambda^*(g(C(\{y_i\}) \cup C(\{z_i\}))) \geq \lambda^*(g(E)).$$

<sup>3</sup>While the statement of Besicovitch's Covering Theorem is usually given in  $\mathbb{R}^d$  and with only rough bounds for the number of sequences necessary, it is not too hard to show that two sequences suffice for  $\mathbb{R}^1$ .

Since  $\epsilon$  was arbitrarily small,  $\lambda(g(E)) = 0$ .  $\square$

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